Outline of Topology
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MATH>Topology

topology

Topology is about non-metric space properties {topology}| {analysis situs}.

properties

Open-set space properties are continuity, proximity, homeomorphy, and invariance under transformation {topological property}. Topological properties are invariant if figures bend, stretch, shrink, or deform without creating or losing points and nearby points are still nearby. Being a curve is invariant topological property. Space dimension is an invariant topological property.

not properties

Size, shape, angle, or length are not open-set space properties and are not topological properties.

atlas in topology

Geometries have intrinsic properties {atlas, topology}|. Curvature is a local intrinsic surface property. Connectivity and orientability are global intrinsic properties.

homeomorphism

Two functions or topologies are equivalent if both are continuous and in one-to-one correspondence {topological equivalence} {homeomorphism}|. Elastic figures are topologically equivalent. Cut figure that reattaches is topologically equivalent to original figure.

MATH>Topology>Curve

curve in topology

Intervals {curve, topology} can map or transform onto planes. Curves are positive-direction moving-point paths.

rectifiable curve

Curves {rectifiable curve} can have limiting arc lengths. For interval (a,b), arc length = integral from x = a to x = b of (1 + (f'(x))^2)^0.5 * dx, where f(x) is curve function, and f'(x) is derivative.

winding number

Curves can wind 360 degrees around plane points {winding number} {index of the mapping}, depending on surface topology.

MATH>Topology>Interval

interval in topology

Point sets {interval, topology} can connect. Neighborhoods are open intervals. Unbounded intervals have no upper, no lower, or no upper or lower bound.

accumulation point

Open sets around points {accumulation point, topology} contain sequence points.

bounded interval

Functions can have all points inside n-sphere {bounded interval}.

compactness in topology

Functions can be in closed and bounded intervals {compactness, topology}|. Intervals can have limits for all infinite subsets. Non-compact manifolds have edges or are infinite.

completeness in topology

Curves, surfaces, spaces, or functions can have no breaks or missing values over real-number intervals {completeness, topology}|.

connectedness in topology

Curves, surfaces, spaces, or functions can have no breaks or missing values over real-number intervals {connectedness, topology}|. All convex figures connect. Interval boundaries are in two disjoint subsets. Intersections are never empty.

continuous in topology

All points can have neighborhoods {continuous, topology}. Functions have inverses.

continuum in topology

Curves, surfaces, spaces, or functions can be in closed intervals and have connectedness {continuum, topology}. Curves are one-dimensional continuums, closed at infinity. Sphere surfaces are two-dimensional continuums.

disjoint set in topology

Two sets {disjoint set, topology} can share no points. Two disjoint open sets have disjoint closed sets inside {normal space}.

neighborhood of interval

Points can be in sphere-like regions {neighborhood}|. Neighborhoods are open sets.

osculating circle

At curve or surface points, circles {circle of curvature} {osculating circle} can have radius equal to curvature radius.

MATH>Topology>Interval>Boundary

closed interval

Intervals {closed interval, topology} can have all points within or on higher boundary {upper bound, interval} and within or on lower boundary {lower bound, interval}.

open interval

Intervals {open interval, topology} can have all points within highest value {least upper bound, interval} and within lowest value {greatest lower bound, interval}. Neighborhoods are open intervals.

MATH>Topology>Polyhedron

combinatorial topology

Geometric-figure topologies {combinatorial topology} {algebraic topology} can be mathematical groups. Face, edge, and vertex numbers can relate.

cross cut

If surfaces have edges, closed curves can go from one edge to the other {cross cut}.

Euler law

For closed convex polyhedrons, vertex number v minus edge number e plus face number f equals two: $v - e + f = 2$ {Euler's law} {Euler law}.

network in topology

Topologies {network, topology}| can have nodes {vertex, network} connected by lines {edge, network}. Two non-parallel surfaces intersect at an edge. Three non-parallel surfaces intersect at a vertex. Non-zero areas intersect at edges {path arc} {arc, boundary} {boundary, arc} and vertices and are finite in number.

connected

Networks can have one or more vertices that have only one edge. Networks {connected network} can have at least two path arcs at all vertexes.

map

Networks {map, network} can have path arcs that bound countries or regions.

path

Traveling along edges leads to vertex sequences {path}. Traveling around whole network can involve returning to vertex or never returning.

random

Keeping node number constant, nodes can have random numbers of random links to other nodes {random network} {exponential network}. Nodes have same average link number. Probability that node connects to N other nodes decreases exponentially with N. Variance can be small {Poisson distribution, network}.

hubs

Not keeping node number constant, random networks {scale-free network} can have nodes {hub} with more links than others. Probability that node connects to N other nodes is approximately proportional to (1/N)^2. Early nodes tend to have more links. Several nodes are critical. Affecting single and many random hubs is unlikely to break network. Internet, social networks, alliances, and cell biochemical reactions can be scale-free networks.

point set topology

Geometric-figure or space topologies {point set topology} can be points related by distance functions or limits.

topological graph

Figures {topological graph} can have nodes and connection paths between nodes. Nodes with even number of paths are even. Nodes with odd number of paths are odd.

triangulation of polygons

Two-dimensional polygons can change into triangles {triangulation, polygon}.

MATH>Topology>Polyhedron>Complex

complex in topology

Finite linear simplex combinations {complex, topology} can meet or not meet in common faces. Complexes have values {characteristic, complex} found by formulas {Euler-Poincaré formula}. Complexes {chain, simplex} {simplex chain} can be linear oriented-simplex combinations. Boundary of chain boundary equals zero.

Poincare group

Space-time can have transformations. For four-dimensional flat space-time, ten transformations leave proper time (and proper length) between two events (with a trajectory between them) unchanged {isometry}: translation through time dimension, translation through space dimension 1, translation through space dimension 2, translation through space dimension 3, fixed-angle rotation around space dimension 1, fixed-angle rotation around space dimension 2, fixed-angle rotation around space dimension 3, no-rotation velocity change (Lorentz transformation) {boost} along space dimension 1, no-rotation velocity change along space dimension 2, and no-rotation velocity change along space dimension 3. Series of these transformations also leave proper time (and proper length) unchanged. Therefore, this transformation set forms a group {Poincaré group}, which shows rotational symmetries of empty space and special-relativity time coordinates.

The Poincaré group is about Minkowski space-time isometries and is a ten-dimensional noncompact Lie (abelian) group. The Poincaré group defines Minkowski-space-time geometry and is the relativistic-field-theory group. In quantum mechanics, particle mass (four-momentum), spin, parity, and charge are positive-energy unitary irreducible Poincaré-group representations.

In topology, complexes have Poincaré groups {first homotopy group}. Group operation traverses curve and then traverses another curve in same direction. Curves that can deform into each other have one class and are homotopic. Therefore, chain and cycle theory and group theory have equivalences.

For three-dimensional manifolds, two simplex subdivisions {Haupvermutung} are isomorphic. At least one singular point exists on even-dimensional spheres.

MATH>Topology>Surface

cycle in topology

If surface has more than two dimensions, closed surfaces {cycle, topology} exist.

orientable

Closed surfaces can have orientation.

simply connected

Closed surfaces can have no singularities or infinities.

independence

For closed surface, chain can meet itself, so boundary equals zero. Cycles are dependent if chain boundary is not zero. Number {Betti number} of manifold-dimension independent cycles is invariant. In orientable n-dimensional cycles, dimension-p Betti number equals dimension n - p Betti number {duality theorem}.

torsion

Number {torsion coefficient} of times cycle {torsion cycle} must multiply to bound is invariant. Simply connected closed n-dimensional manifolds are homeomorphic to spheres with same Betti number and torsion coefficients. Spheres are simply connected closed two-dimensional manifolds.

Gaussian coordinate

Meshes or nets can be on surfaces. Scale can vary from point to point {Gaussian coordinate} {curvilinear coordinate}, in which case magnitude is not important. Locally, surface manifolds can have curvature zero, be flat, and use Pythagorean theorem to find distances, vectors, and metric.

geometrization

Topology can have geometry {geometrization}, which has constant curvature.

Ricci flow equation

Like heat flows from hot to cold and makes uniform temperature, curvature can flow to make constant curvature {Ricci flow equation}. However, Ricci flow allows singularities, with different starting geometries. For example, dumbbell shape tends to make two spheres with point between, rather than one large sphere. Thin rod tends to have singular point at one end {cigar singularity}. If sphere replaces singularity, Ricci flow can continue. Ricci flow can find possible shapes in compact spaces (Richard Hamilton) [1982].

Grigory Perelman [2003] used Ricci flow to show that all Ricci-flow-procedure singularity types can morph into spheres or tubes in finite time, so procedures can remove them from space.

Riemann-Roch theorem

Two theorems {Riemann-Roch theorem} {residual theorem} are about regular and irregular surfaces.

MATH>Topology>Surface>Genus

genus of surface

Connectivity and closedness combine to make surface property {genus, surface}|. Closed Riemann surfaces with same genus are topologically equivalent. Genus is invariant under birational transformation.

sphere

Sphere has genus 0. Genus-0 closed surfaces can map onto spheres and connect simply. Sphere projective planes have genus zero, are closed, and look like circles with semi-circumference line at infinity.

handles

Sphere with n handles has genus n. The one-sided-surface Klein's bottle has genus one, because it has one handle.

holes

Sphere with n holes has genus n. Hole number equals function branch-point number divided by two, minus function-value number, plus one. For curves, genus equals 0.5 * (n - 1) * (n - 2) - d, where d equals double-point number and n equals function degree. Riemann surface corresponding to genus-p curve has connectivity 2*p + 1.

connectivity of surface

Number of possible closed curves {connectivity, topology}| can differ. Connectivity is number of closed surfaces that do not make disjoint regions. Spheres have one loop type. Toruses can have loops around and loops across. Closed curves can follow surface in different ways. Loops {loop out} completely bound closed surface. Connectivity is a global surface property.

homotopy

Surface-topology indexes {homotopy, topology}| can measure how many ways a closed curve can be in a surface. First way is that loop can become point, as on sphere. For other topologies, loop cannot become point. Second way is that loop can become circle, as across torus. Loops and mathematical groups are homotopic. Functions or structures have symmetries.

uniformization theorem

For all genuses, using parameters can make multiple-valued functions into single-valued functions {uniform function} {uniformization problem} {uniformization theorem}. For genus zero {unicursal curve, genus}, parameterized function f(w,z) can equal zero. For genus equal one {bicursal curve}, parameterized function f(w,z) can equal zero.

MATH>Topology>Surface>Manifold

manifold in topology

Curves or curved surfaces {manifold, topology}| can have Euclidean-space neighborhoods at all points, and neighborhoods connect continuously at overlapping open regions. Point or element sets are continuous functions specifiable by coordinates. Manifolds are generalized Riemann surfaces.

examples

Planes, spheres, and toruses are two-dimensional manifolds.

topological equivalence

Two two-dimensional manifolds can be topologically equivalent. For three-dimensional manifolds, the proof does not yet exist that any two manifolds can be topologically equivalent. For four-dimensional manifolds, no algorithm can prove that any two manifolds can be topologically equivalent.

metric

Riemannian manifolds can have a metric.

boundary

Manifolds with boundaries have neighborhood part on boundary.

Hausdorff space

All points can have open sets that do not intersect, so there is no branching {Hausdorff space}. Two distinct points have open sets that do not intersect other open set.

Volume depends on line element raised to power {Hausdorff dimension} and is not necessarily an integer.

fields: scalar

Manifolds can have smooth coordinate functions, making scalar fields. Manifolds are commutative scalar-field algebras.

fields: vector

Manifolds can have differentiation operator on scalar field, making vector field.

fields: covector field

Manifolds have symmetrical duals {1-form, topology} to vector fields {covector field}. Covector space is an n - 1 dimension hyperplane.

Vector spaces have tangents at points. Tangents have duals {covector space}.

p 1-form intersections make dimension n - p hyperplanes {p-form} {simple p-form}. p-forms are integrable {exterior calculus, manifold} to find density or gradient. Exterior derivative gradient is covector space.

Vector-field tensor differentiation depends on tangent vectors {covariant derivative operator} {connection, derivative}.

curvature

Curvature tensor measures vector change after parallel transport around loop.

torsion

If there is no torsion, curvature tensor is zero {first Bianchi identity, torsion} {Bianchi symmetry, torsion}. If there is no torsion, curvature-tensor derivative is zero {Bianchi identity, torsion} {second Bianchi identity, torsion}. Torsionless connections {Riemannian connection} {Christoffel connection} {Levi-Civita connection} can preserve vector length during parallel transport.

2-manifold

Non-metric spaces {2-manifold} can have two dimensions. Three geometrized 2-manifolds exist, depending on curvature. The simplest compact 2-manifold is a sphere with positive curvature. Torus is a 2-manifold with zero curvature. 2-manifolds can have negative curvature when they have two or more handles.

3-manifold

Non-metric spaces {3-manifold} can have three dimensions. 3-manifolds are possible three-dimensional topologies. The simplest compact 3-manifold is a 3-sphere. Other 3-manifolds have edges or have multiple paths from one region to another. People know all 3-manifold types. Manifolds can have orientation or not.

Thurston geometrization conjecture

Eight geometries can make geometrized 3-manifolds {Thurston geometrization conjecture}: positive curvature, negative curvature, flat curvature, and five 2-sphere combinations with curvatures.

Poincare conjecture

3-spheres are unique 3-manifolds {Poincaré conjecture} [1904]. If all loops in three-dimensional compact closed space can shrink to points, without breaking loop or space, space is 3-sphere, and no other such 3-space exists. 4-spheres are unique 4-manifolds, as proved by Michael Freedman [1983]. 5-spheres are unique 5-manifolds, and all higher dimensional spheres are unique manifolds, as proved by Stephen Smale [1960].

MATH>Topology>Surface>Orientation

orientation in topology

Topology can define direction {orientation, topology}. To turn bounded surfaces inside out, project at right angles through plane defined by surface boundary line, to make projected-figure points same distance from plane as original bounded surface {inside out}. Turning figures with surface arrows inside out reverses arrow direction.

orientability

After one loop, vector direction is either same or opposite {orientability}|. Orientability is a global property.

orientable surface

On triangulated surfaces {orientable surface}, triangles can orient so sides common to two triangles orient oppositely on two triangles. Spheres are orientable surfaces. Projective planes are not orientable surfaces. If and only if two orientable closed surfaces have same genus, surfaces are homeomorphic.

version on surface

If arrow goes around Klein bottle or Möbius strip once, arrow reverses direction {version, topology}|, the same as turning figure inside out.

MATH>Topology>Problems

topological problems

Topology can determine if equation or simultaneous equations have solutions {topological problems}. Function must be continuous, compact, and connected. If winding number is greater than zero, two simultaneous equations have solutions.

Konigsberg bridges problem

Can one cross all seven bridges of old Königsberg only once and return to start {Königsberg bridges problem}? Königsberg had seven bridges and two river islands.

Euler replaced bridges by lines and land by points to show that one path cannot traverse the bridges. Königsberg bridges problem is a topological graph, in which land is nodes and bridges are connections. If all nodes are even, one can cross all bridges once and return to start. If one or two nodes are odd, one can cross all bridges once but end at another point. If more than two nodes are odd, one must cross at least one bridge more than once.

tower of Hanoi

Games {tower of Hanoi} can use a horizontal board with three vertical pegs. Discs are on one peg, with largest on bottom. The idea is to transfer discs to another peg to recreate same series, without ever putting a larger disc on a smaller-disc top. With n discs, it requires 2^n - 1 transfers.

traveling-salesman problem

Salesman wants to travel shortest distance among cities, with no path duplication {traveling-salesman problem}.

MATH>Topology>Problems>Map

four-color theorem

Four colors can color two-dimensional maps so no two countries with common boundary have same color {map problem} {four-color theorem}|. Kenneth Appel and Wolfgang Haken proved the four-color theorem [1976], by enumerating possibilities and then checking all by computer. One-dimensional line maps need two colors. Three-dimensional maps need six colors.

MATH>Topology>Topologies

Klein bottle

One-sided-surface bottles {Klein's bottle} {Klein bottle} have genus one, because it has one handle.

Mobius strip

One-sided surfaces {Möbius strip}| {Möbius band} can exist. Normal that moves around surface axis from point and returns to the point has opposite direction. If it goes around again, it has original direction.

polyhedron in topology

Two-dimensional-polygon sets {polyhedron, topology} can change into triangles, or triangle sets can change into polygons.

compact space in topology

Two-dimensional spaces {compact space} can be planes, spheres, or saddles.

Thurston conjecture

Three-dimensional compact spaces have eight basic shapes {Thurston conjecture} (William Thurston) [1985].

wallpaper group

Plane has 17 figure symmetries {wallpaper group, topology}.