Outline of Group Theory
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MATH>Group Theory

group in mathematics

Element sets {group, mathematics} {mathematical group}, such as equilateral-triangle points, can have one or more operations, such as rotations through 60-degree angles. All elements are products {word, group} of the operation {generator, group} acting on group elements.

elements

Elements can be geometric or algebraic. Groups have finite or infinite number {order, group} of elements.

operation

Operation maps element to element {product, operation}. For example, rotation maps point to second point. Addition maps two elements to one element {binary operation}.

generator

Some groups have constraints on generators, and some groups (free group) have no constraints on generators.

identity

Operations can map element to same element {identity operation}.

inverse

Operations can be another operation's opposite {inverse operation}. Operation followed by inverse operation makes original element.

association

For three elements and a binary operation, do operation on first and second elements, and then do operation on product element and third element, (a + b) + c, or do operation on second and third elements, and then do operation on first element and product element, a + (b + c). The resulting products can be the same (associative group) {association, group} or different {non-associative group}.

commutation

Binary operations can happen in either order: a + b or b + a. For two elements and a binary operation, do operation on first and second elements, a + b, or do operation on second and first elements, b + a. The resulting products can be the same (commutative group) {commutation, group} or different {non-commutative group}. Non-commutative operations typically make the negative: a + b = -(b + a). Complex-number multiplication is commutative. Matrix multiplication is non-commutative.

transformation groups

Different spaces and geometries have different invariant transformations. For example, different spaces and geometries can be invariant under rigid motions {motion geometry}. For different space geometries, number of dimensions determines number of possible shape changes. Space geometries are in a hierarchy from least general to most general: metrical geometry, affine geometry, projective geometry, and topology.

Transformation groups have operations that are linear transformations {analytic transformation, group}. Different groups have different invariant transformations. Transformation groups describe spaces and geometries. For example, group U(1) describes circle rotation symmetries (electromagnetism has U(1) symmetry, which implies charge conservation). Symmetry groups {universality class} represent substances with different symmetries.

Transformation results can be in tables {multiplication table}. Rows and columns are elements, and cells have products.

types

Transformation groups are finite continuous groups. Differential equations make an infinite continuous group. Substitution groups are finite-order discontinuous groups. Automorphic functions are infinite-order discontinuous groups.

matrix

Matrices can represent groups.

group theory

Mathematical groups have invariance types {group theory}. Groups with same invariance type are mathematically the same. Group theory is complete, because mathematics has discovered all invariance types. Finite-group theory is not isomorphic to any other theory. Equation theory, number theory, and infinite-group theory are mathematically equivalent.

Burnside problem

For finitely generated groups, if elements have finite order, is group finite {Burnside's problem} {Burnside problem}?

class of group

Groups can split into mutually exclusive sets {class, group}. One class is the identity class. Classes {conjugate class} can have other-class conjugates. For group A and its conjugate A^-1, if A^-1 * B * A = C, B and C are conjugates, and A * C * A^-1 = B.

isomorphism problem

When are two groups isomorphic {isomorphism problem}?

Lagrange theorem

Subgroup order divides finite-group order {Lagrange's theorem} {Lagrange theorem}.

Liouville theorem

Phase-space volume is constant {Liouville's theorem, group} {Liouville theorem, group}.

MATH>Group Theory>Field

field

Groups {field, mathematics} can have two operations, such as adding and multiplying. For example, rational numbers and real numbers form mathematical fields over addition and multiplication. The rational-function field (which includes polynomials) has adding and multiplying operations. Rational numbers, real numbers, and rational functions are infinite fields. Other fields are algebraic number fields (algebras), relative Galoisian fields (Galoisian equation), relative Abelian fields (polynomial equations), class fields, valuation fields, separable fields, and prime fields.

p-adic field

Fields {p-adic number} {p-adic field} can have sums, from -p to infinity, of rational number times prime numbers.

MATH>Group Theory>Group Theories

Abelian equation

For polynomial equations {Abelian equation}, all roots are square roots or first-power roots. Square roots and first-power roots can interchange.

Galois theory

Some polynomial equations are solvable by algebraic operations {Galois theory}, and mathematical groups show which types.

For equations with prime-number degree {Galoisian equation}, solutions are rational functions of two roots, found from two linear equations, two quadratic equations, or one linear and one quadratic equation.

To find polynomial roots, find successively smaller groups down to all smallest normal subgroups {composition series}. The set of composition-series subgroups is unique. If composition-series subgroup indices are prime numbers, roots have radicals.

Quotient groups do not depend on composition-series subgroups {Jordan-Holder theorem}.

renormalization group

To understand critical phenomena, use reiterative group theory {renormalization group theory}. Critical phenomena include Curie point, critical temperature, critical pressure, superconductivity transition temperature, and alloy metal-atom order-disorder temperature. Renormalization-group-theory operations eliminate infinities by reiteration to eliminate higher-order terms.

representation theory

The non-commutative GL(n) subgroup has matrix operations {representation theory}.

MATH>Group Theory>Mathematical Groups

Abelian group

Groups {Abelian group} {commutative group} can have commutative operation.

associative group

Groups {associative group} can have associative operation.

continuous group

Groups {continuous group} can have local operations on infinitesimal elements, to make real-number algebras.

Lie group

Continuous commutative or non-commutative groups {Lie group}, such as n^2 parameter Lie groups {SU(n) group}, have parameters and make associative and non-associative algebras (Lie algebra).

non-Abelian group

Groups {non-Abelian group} can have non-commutative operations.

orthogonal group

Classical groups {symmetric bilinear form} {orthogonal group} can preserve linear vector transformations in three-dimensional Euclidean space, because axes stay at right angles and maintain non-singular quadratic form.

matrices

Matrices can represent orthogonal groups. For orthogonal-group transformations, inverse matrix equals transpose matrix.

Matrix diagonal has 0, 1, or -1. Orthogonal groups {Lorentz group} can have diagonal {Lorentzian diagonal} with one 1, zero 0s, and non-zero -1s. Number of 1s and -1s can be invariant {signature, group}.

If diagonal has only 1s {positive-definite diagonal}, matrix is non-singular and positive, and has positive-definite non-singular symmetric tensors {metric, tensor}. Group {pseudo-orthogonal group} matrix can be singular and not positive-definite {pseudometric}.

If orthogonal-group transformation determinant equals 1, group is non-reflective.

permutation group

Groups {substitution group} {permutation group} can have operation that maps one element to another element {substitution, group}. An example is rearranging element positions {permutation, group}, such as exchanging elements at line-segment ends. Permutations can result in equivalent subsets {combination, group}. Product element is in group {closure, group} {closed group}.

identity

Substitution groups must have identity operation.

inverse

Substitution-group operations must have inverse operations.

transitive

Group operation, such as rotations, can substitute all elements {transitive group}. Group operations, such as division, may not apply to all elements or some elements may not be operation results {intransitive group}.

primitive

Elements can have separate subsets that permute among themselves {imprimitive group} or have no separable subsets {primitive group}.

abstract groups

Generalized substitution groups {abstract group} show group properties.

symmetry group

Groups {symmetry group, mathematics} can have operations that leave system unchanged {symmetry transformation}.

types

If elements are alphabet characters, groups {symmetric group} can have all letter permutations, or groups {alternating group} can have all even permutations. If elements are numbers, groups {cyclic group} can have all element powers. If elements are points, motions can go around angles {rotation symmetry}, can be axis linear motions {transformational symmetry}, or can be line motions {translation symmetry}. Transformations {displacement, transformation} can preserve geometric-figure congruence.

types: reflection

If elements are points, motions in planes can go through axes {reflection symmetry}. Reflections can be through points {radial symmetry reflection} {central symmetry reflection}, lines {axial symmetry reflection}, or planes {plane symmetry reflection}. If elements are points, motions can have both rotation and reflection {inversion symmetry}.

symplectic group

Classical groups {symplectic group} can have linear transformations with anti-symmetry (transpose equals negative of original).

topological group

Groups {topological group} can be continuous, infinite, connected, and compact.

transformation group

Groups {transformation group} can be finite continuous groups.

unitary group

Classical groups {unitary group} can preserve complex linear transformations and maintain non-singular Hermitean form, rather than orthogonal-group quadratic form. Hermitean form equals its complex conjugate {Hermitean conjugation}, which is the dual space. For unitary groups, linear-transformation inverse is positive-definite transpose. For pseudo-unitary groups, linear-transformation inverse is not positive-definite transpose.

MATH>Group Theory>Mathematical Groups>Morphism

homomorphism of group

Smaller groups {homomorphism, group} can have same elements and same multiplication table as larger-group subgroup. For example, C4v group is homomorphic to group of multiplications of 1 and -1. Several extended-group members can associate with same smaller-group member {many-to-one mapping}. Isomorphism and homomorphism are independent group properties.

isomorphism of group

Two groups can have same number of elements and same multiplication table {isomorphism, group}. For example, group of eight 2x2 matrices with elements 0 and 1 is isomorphic to C4v group of square symmetries. Isomorphism and homomorphism are independent group properties.

MATH>Group Theory>Mathematical Groups>Subgroup

subgroup

Groups can have smaller groups {subgroup}. If subgroups share only the identity element, and multiplication commutes, products of two subgroups are subgroups. Groups {composite group} can have invariant subgroups.

coset of group

Group elements can be either non-subgroup elements or subgroup elements. All non-subgroup elements form a set {coset, group}.

factor group

Groups {factor group} can have normal subgroups that factor the group commutatively.

normal subgroup

Groups {normal subgroup, commutation} {invariant subgroup} {self-conjugate subgroup} can have same products when using group member and then subgroup member, or subgroup member and then group member, so group elements commute with subgroup elements.

proper subgroup

Subgroups {proper subgroup} can have more than one element and less than all elements.

quotient group

Subgroup {quotient group} elements can be an invariant-subgroup's cosets.

simple group

Groups, such as complex numbers, can have subgroups. Groups, such as empty set and sets with one element, can have no subgroups. Non-Abelian Lie groups {simple group} can have no normal subgroups.

semi-simple group

Non-Abelian Lie groups {semi-simple group} can have no Abelian normal subgroups.

classical group

Continuous simple groups {classical group} have four families and exceptional groups.

families

For m = 1, 2, 3, ... SU(m + 1) is unitary {Am family}, with dimension m * (m + 2). SO(2*m + 1) is orthogonal {Bm family}, with dimension m * (2*m + 1). Sp(m) is symplectic {Cm family}, with dimension m * (2*m + 1). SO(2*m) is orthogonal {Dm family}, with dimension m * (2*m - 1).

exceptional groups

Other continuous simple groups {exceptional groups} are E6, E7, E8, F4, and G2.

finite

Finite simple groups have classical groups and exceptional groups. SO(3) group is Special, because it has unit determinant and so is non-reflective. SO(3) group is Orthogonal, because the three axes are at right angles. SO(3) group has number 3 because rotations can be in three dimensions.

product

Simple groups can combine {product group} to make element pairs.

product: linear

n-dimensional vector spaces have product groups {general linear group} GL(n) of linear translational symmetries. GL(n) is product group of n x n singular matrices, one for each dimension.

sporadic group

Simple groups {sporadic group} can have finite number of elements, such as Janko J4, Fischer Fi24, Baby Monster B, Monster M, M12, M24, and Co1. There are 26 sporadic groups.

MATH>Group Theory>Ring

ring

Fields {ring, field} can have only addition, subtraction, and multiplication, with no division. Rings have distributive property. Rings can have unit element or not. In rings, one operation is commutative, and other operation is closed, associative, and not necessarily commutative. If rings have unit element and inverse elements, they are non-commutative {division ring}. Rings have elements {center, ring} that do commute. Ring elements {kernel, ring} can correspond to another ring's zero elements, by homomorphism.

Lie algebra

Rings can be associative and non-associative algebras {Lie algebra}.

Noetherian ring

Commutative rings {Noetherian ring} can have ideals in forms (Amalie Noether or Emmy Noether).

MATH>Group Theory>Selection Operation

selection operation

Group operation {selection operation} can select from group. All selections can have equal effect {idempotent}. Two selected members can have nothing in common {non-overlapping}. Selecting all members can equal selecting whole set {exhaustion}.

spectral set

Selection can be idempotent, non-overlapping, and exhaustive {spectral set}.

MATH>Group Theory>State Space

state space

Spaces {state space}| can have possible states and an operation on one or more states that makes a state. States are n-dimensional point sets (vectors), as in Hilbert spaces.

metric

Spaces allow measurements. Measurability has rules {convergence rule}.

trajectory

Spaces can have initial states and goal states, which link along intermediate-state paths (trajectory). Distance metric, matching features, or total optimal value heuristic search can find trajectory.

spaces and groups

The possible states are elements, so the state-transition operation makes a group. Transformations represent forces and motions.

matrix

Matrices {second-order tensor} can describe states and operations, which operate on vector states to change state vectors or find vector interaction results. Matrix transformations form groups.